Analytic Determination of the Capacitance Matrix of Planar or Cylindrical Multiconductor Lines on Multilayered Substrates

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***Abstract*—An exact analytical expression for the Maxwell capacitance matrix of a multilayer, multistrip planar or cylindrical line is derived by solving the dual series equation system of the problem by means of a Volterra boundary-value problem. The solution is expressed in terms of some infinite matrices with very good convergence properties. Numerical examples show that the method yields accurate results and is also computationally effective for lines having a large number of conductors.**

I. Introduction

Recent advances in integrated circuit technology have made multiconductor transmission lines an important component not only in microwave and millimetre-wave applications, but also in high-speed digital IC's. Although an accurate treatment requires the full wave analysis, the quasi-TEM solution is a good approximation at low frequencies; moreover, the solution obtained by means of the quasi-TEM approximation can be taken as the basis for solving the full propagation problem.

The literature on this subject is extensive; in order to emphasise the different mathematical techniques used, we briefly review some theoretical methods.

Wheeler [1] used approximate conformal mapping and an interpolation technique to calculate the capacity of an inhomogeneous microstrip. Analytical or quasi-analytical solutions have been provided for a limited number of cases and for certain particular geometries [7]–[11]. On the other hand, several numerical approaches based on Green's function integral formulations of the problem have been proposed for the analysis of microstrip structures. Bryant and Weiss [2] treated the dielectric vacuum boundary by means of a dielectric Green's function. Yamashita and Mittra [3] presented an analysis based on a variational principle. Analysis of various planar transmission lines have been carried out in the spectral domain by Itoh and Mittra [4]. The spectral domain method was applied to the more general case of a multiconductor line in [5], [6]. Medina and Horno [15] developed quasi-analytical methods for speeding up the evaluation of spectral series.

Recently, several authors approached the problem of the exact analysis of microstrip structures. Thus, Fikioris *et al.* [12] have given an accurate quasi-TEM study of the boxed microstrip line printed on an isotropic substrate based on the regularization of the Carleman's integral equation. In [13] an analytical solution is given to the microstrip problem based on a special representation formula resulting from some complex-variable boundary-value problems. In [14], the same method is applied to the full wave solution of the microstrip problem.

This paper provides an analytical determination for the capacitance matrix of planar or cylindrical multiconductor lines embedded in a multilayered medium. The analysis method is developed for a cylindrical structure; planar structures can be readily analyzed, since they are amenable to cylindrical structures by means of an intermediate conformal mapping. The solution is exact, but it is expressed by means of some infinite matrices. These matrices have good convergence properties; this confers to the method very attractive features. The analysis is based on the reduction of the singular parts of the series equations of the problem to a Volterra boundary-value problem. The existence conditions for the solution of this boundary-value problem yield the desired capacitance matrix, and the analysis can be also adopted to compute the electric field inside the structure; in this case, it needs also a method for speeding-up the convergence of the infinite series involved in the field expression.

Section II describes the geometry of the cylindrical multiconductor line and how the planar line structure is mapped into it. In the next section, the cylindrical multiconductor line problem is formulated and its solution is reduced to a system of dual series equations. In Section IV the mentioned system of series equations is transformed by using some boundary-value problems in complex plane into an infinite system of algebraic equations. This also yields the Maxwell capacitance matrix of the line, expressed by means of some infinite matrices. The case of strips placed on different cylindrical surfaces and that of multilayered multiconductor structures are considered in Sections V and VI. Numerical examples provided in Section VII show how the method applies to the

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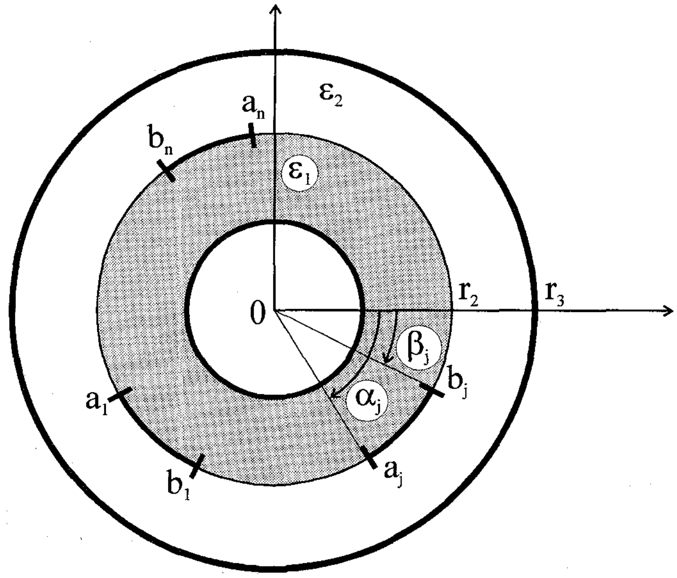

Fig. 1. Two-layer cylindrical multiconductor line.

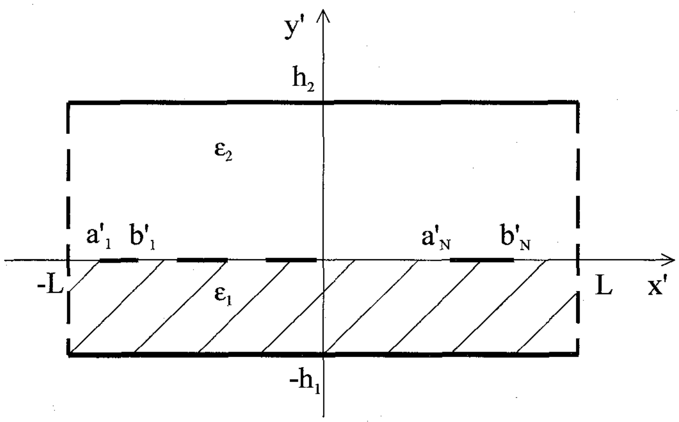

Fig. 2. Cross-section of a planar structure cell.

analysis of different practical line structures: a single strip or a pair of strips inside a rectangular box, a system with many coupled strips, a suspended line, and some multilayer structures.

II. Planar Lines and Cylindrical Lines

It is well known that one of the methods for studying the cylindrical multiconductor stripline transmission lines, operating into a quasi-TEM mode, is to transform them by a suitable conformal mapping into an equivalent planar structure [11], [21], [22]. Furthermore, the capacity matrix of the cylindrical structure can be expressed by means of the Maxwell capacitance matrix of the equivalent planar structure.

In this paper we reverse the procedure: first we put into evidence an efficient method for determining the capacitance matrix of a cylindrical structure and, further on, we characterise the planar equivalent structure by using this matrix.

We begin the analysis with a two-layer structure, a more general case will be addressed in Section V. The cross-section of the cylindrical multiconductor transmission line to be analysed is shown in Fig. 1. It consists of three surfaces S_1, S, S_2; the cylindrical surfaces S_1 and S_2 (of radii r_1 and r_2, respectively) are grounded and the circuit surface S separates two different dielectric media of relative dielectric constants $\varepsilon_1, \varepsilon_2$. On the circuit surface are placed N conducting strips of zero-thickness $[a_j, b_j]$, $(j = 1, \ldots, N)$, with arbitrary widths and spacing. We normalize all linear dimensions with respect to the radius of the circuit circle and we characterize the electrodes position by the angles α_j, β_j, $(j = 1, \ldots, N), -\pi < \alpha_1, \beta_N < \pi$. Let $a_j = \exp(i \cdot \alpha_j), b = \exp(i \cdot \beta_j), (j = 1, \ldots, N), \alpha_{N+1} = \alpha_1 + 2 \cdot \pi$. We denote by D_1 the domain filled with the medium of dielectric constant ε_1 and by D_2 the other domain.

Let us consider now the planar structure in Fig. 2. It consists of N conducting strips of zero thickness placed on a dielectric substrate between two parallel ground planes. We assume the structure as being a cell of a periodic infinite system, hence the boundary conditions on the lateral sides of the box result from periodicity. By means of the conformal mapping

$$z = \exp\left(-i\frac{2\pi}{L}z'\right), \tag{2.1}$$

the considered structure is mapped into the cylindrical structure of Fig. 1. with parameters:

$$r_1 = \exp\left(-\frac{2\pi}{L}h_1\right), \quad r_2 = \exp\left(\frac{2\pi}{L}h_2\right),$$
$$\alpha_j = -\frac{2\pi}{L}b'_{N-j+1}, \quad \beta_j = -\frac{2\pi}{L}a'_{N-j+1}. \tag{2.2}$$

As the capacity matrix is preserved by a conformal mapping, it results that the Maxwell capacitance matrix of the system in Fig. 2 is the same as the capacitance matrix of the system in Fig. 1.

In the case where we have a structure consisting of p conducting strips placed on a dielectric substrate inside a ground box, we must have only electric walls on all the box sides. This is why we consider, for the beginning, the symmetrical structure given in Fig. 3, but with antisymmetrical potentials. This assures an electric wall along the Oy' axis. Let $C^{(2p)}$ be the capacity matrix of the periodic $2p$ structure given in Fig. 3. Then, the capacitance matrix of the boxed p-strips structure is given by relation

$$C_{i,j}^{(e)} = C_{i,j}^{(2p)} - C_{i,2p-j+1}^{(2p)}, \quad i = 1, \ldots, p. \tag{2.3}$$

By taking a symmetrical charge on the strips of the structure in Fig. 3, we can obtain also the capacity matrix of the structure consisting of n strips placed inside a box with magnetic lateral sides in the form

$$C_{i,j}^{(h)} = C_{i,j}^{(2p)} + C_{i,2p-j+1}^{(2p)}, \quad i = 1, \ldots, p. \tag{2.4}$$

III. Analysis of the Cylindrical Structure

We consider the solutions of the cylindrical multiconductor transmission line problem in the "quasi-static" approximation, i.e. for the frequency range in which propagation may be regarded as quasi TEM. Under quasi-TEM approximation, the analysis of a coupled microstrip line with a nonmagnetic

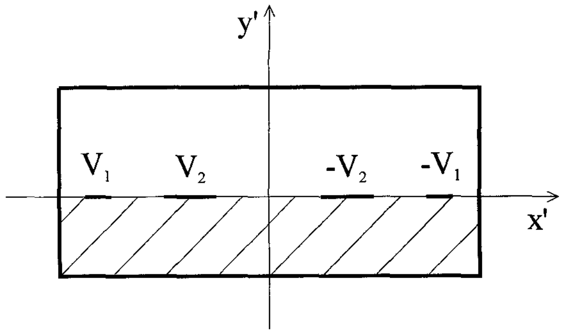

Fig. 3. Planar structure with antisymmetrical excitation used to analyze the boxed line.

material is reduced to that of the capacitance matrix per unit length.

In the quasi-TEM state, the electric field in domains D_1 and D_2 can be expressed with the aid of the electrostatic potentials $V^{(1)}(x, y), V^{(2)}(x, y)$.

By the method of the separation of the variables in cylindrical coordinates, (r, ϕ) $V^{(1)}$ and $V^{(2)}$ can be generally expressed as

$$V^{(j)}(r,\phi) = \Gamma R_0^-(r, r_j, 1) + \sum_{n=1}^{\infty} R_n^-(r, r_j, 1)(A'_n \cos n\phi + B'_n \sin n\phi), \quad j = 1, 2. \tag{3.1}$$

The two potential functions $V^{(1)}(r,\phi), V^{(2)}(r,\phi)$ are solutions of the Laplace equation in domains D_1, D_2, respectively, satisfying the boundary conditions on the inner and on the ground surfaces $V^{(1)}(r_1, \phi) = 0$ and $V^{(2)}(r_2, \phi) = 0$ and the continuity on the circuit surface

$$V^{(1)}(1,\phi) = V^{(2)}(1,\phi) = \Gamma + \sum_{n=1}^{\infty}(A'_n \cos n\phi + B'_n \sin n\phi). \tag{3.2}$$

In (3.1) we have denoted

$$R_0^-(r, r_1, r_2) = \frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)},$$

$$R_n^{\mp}(r, r_1, r_2) = \frac{\left(\frac{r}{r_1}\right)^n \mp \left(\frac{r_1}{r}\right)^n}{\left(\frac{r_2}{r_1}\right)^n - \left(\frac{r_1}{r_2}\right)^n}. \tag{3.3}$$

The real constants $\Gamma, A'_n, B'_n, (n = 1, \ldots, \infty)$ will be determined by imposing the remaining boundary conditions on the circuit surface. Thus, we must have

$$\Gamma + \sum_{n=1}^{\infty}(A'_n \cos n\phi + B'_n \sin n\phi) = V_j \quad \text{for } \phi \in (\alpha_j, \beta_j), j = 1, \ldots, N, \tag{3.4}$$

and

$$\rho(\phi) = -\varepsilon_2 \frac{\partial V^{(2)}(1,\phi)}{\partial r} + \varepsilon_1 \frac{\partial V^{(1)}(1,\phi)}{\partial r} = 0, \quad \phi \in (\beta_j, \alpha_{j+1}), j = 1, \ldots, N. \tag{3.5}$$

We denoted V_j the potential of the strip (a_j, b_j) and $\rho(\phi)$ is the surface charge density along the circuit surface. The total charge on the circuit circle on the arc (α_1, ϕ) is

$$\begin{aligned} Q(\phi) &= \int_{\alpha_1}^{\phi} \rho(\phi')d\phi' \\ &= \int_{\alpha_1}^{\phi}\left[\varepsilon_1 \frac{\partial V^{(1)}(1,\phi')}{\partial r} - \varepsilon_2 \frac{\partial V^{(2)}(1,\phi')}{\partial r}\right] d\phi'. \end{aligned} \tag{3.6}$$

Let $U^{(1)}(r,\phi), U^{(2)}(r,\phi)$ be the harmonic conjugate functions, i.e. the flux functions, of the potentials $V^{(1)}(r,\phi)$, $V^{(2)}(r,\phi)$. We have $\frac{\partial V^{(j)}}{\partial r} = \frac{1}{r}\frac{\partial U^{(j)}}{\partial \phi}$ $(j = 1, 2)$, and consequently (3.6) can be written

$$\begin{aligned} Q(\phi) &= \int_{\alpha_1}^{\alpha} \frac{d}{d\phi'}\left[\varepsilon_1 U^{(1)}(1,\phi') - \varepsilon_2 U^{(2)}(1,\phi')\right] d\phi' \\ &= \left(\varepsilon_1 U^{(1)}(1,\phi) - \varepsilon_2 U^{(2)}(1,\phi)\right) \\ &\quad - \left(\varepsilon_1 U^{(1)}(1,\alpha_1) - \varepsilon_2 U^{(2)}(1,\alpha_1)\right). \end{aligned} \tag{3.7}$$

Hence, the boundary condition (3.5) becomes

$$\varepsilon_1 U^{(1)}(1,\phi) - \varepsilon_2 U^{(2)}(1,\phi) = \sum_{l=1}^{j} q_l - \gamma(\varepsilon_1 + \varepsilon_2), \quad \phi \in (\beta_j, \alpha_{j+1}), \tag{3.8}$$

where q_j is the charge on the strip (α_j, β_j) and $\gamma(\varepsilon_1 + \varepsilon_2) = \varepsilon_1 U^{(1)}(1,\alpha_1) - \varepsilon_2 U^{(2)}(1,\alpha_1)$ is a constant.

The Cauchy-Riemann equations in polar coordinates enable us to write the flux functions in the two domains D_1, D_2 as

$$U^{(j)}(r,\phi) = -\frac{\Gamma}{\ln r_j}\phi + \sum_{n=1}^{\infty} R_n^+(A'_n \sin n\phi - B'_n \cos n\phi), \quad j = 1, 2. \tag{3.9}$$

Consequently, the boundary condition (3.8) along the slots becomes

$$\sum_{n=1}^{\infty}(1 + \delta(n))(B'_n \cos n\phi - A'_n \sin n\phi) = -(\varepsilon_1 + \varepsilon_2)^{-1}\left(\sum_{l=1}^{j} q_l - \frac{\phi - \alpha_1}{2\pi}\sum_{l=1}^{N} q_l\right) - \gamma, \quad \phi \in (\beta_j, \alpha_{j+1}), \quad j = 1, \ldots, N, \tag{3.10}$$

where

$$\delta(n) = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\frac{r_1^{2n}}{1 - r_1^{2n}} + \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2}\frac{r_2^{2n}}{1 - r_2^{2n}}, \tag{3.11}$$

and the constant Γ has been expressed in the form

$$\Gamma \equiv \frac{\sum_{l=1}^{N} q_l}{2\pi(\varepsilon_1 + \varepsilon_2)\varepsilon^*},$$

where

$$\varepsilon^* = \left(\frac{\varepsilon_2}{\ln r_2} - \frac{\varepsilon_1}{\ln r_1}\right) \cdot (\varepsilon_1 + \varepsilon_2)^{-1}, \tag{3.12}$$

in terms of the total charge on strips.

The unknown quantities in the representation formulae (3.1) and (3.9) are the coefficients Γ, A'_n, B'_n; the dual series equations for determining them are given by (3.4) and (3.10).

IV. Analytical Approach to the Solution of the Dual Series Equation System and Determination of the Capacitance Matrix

We shall transform the series Equations (3.4) and (3.10) by putting into evidence their singular part. Thus, we first consider the new unknown constants A_n, B_n, given by the relation

$$A'_n - iB'_n = (A_n - iB_n) \cdot (1 - \eta(n)), \tag{4.1}$$

where

$$\eta(n) = \frac{\delta(n)}{1+\delta(n)}, \quad \eta(n) = O(r_1^n + r_2^{-n}), \quad n \to \infty, \tag{4.2}$$

and i is the imaginary unit.

The two series equations become

$$\Gamma + \sum_{n=1}^{\infty}(A_n \cos n\phi + B_n \sin n\phi) = f_j(\phi),$$
$$\phi \in (\alpha_j, \beta_j), \quad j = 1, \ldots, N, \tag{4.3}$$
$$\sum_{n=1}^{\infty}(A_n \sin n\phi - B_n \cos n\phi) = g_j(\phi) + \gamma,$$
$$\phi \in (\beta_j, \alpha_{j+1}), \quad j = 1, \ldots, N, \tag{4.4}$$

where we have denoted

$$f_j(\phi) = V_j + \sum_{n=1}^{\infty} \eta(n)(A_n \cos n\phi + B_n \sin n\phi),$$
$$g_j(\phi) = (\varepsilon_1 + \varepsilon_2)^{-1}\left(\sum_{l=1}^{j} q_l - \frac{\phi - \alpha_1}{2\pi}\sum_{l=1}^{N} q_l\right). \tag{4.5}$$

In fact, as can be seen from (4.2) we separated in the left-hand sides of these relations the singular parts of the series equations. These singular components are responsible for the behaviour of the potential function at the strip edges.

We consider, for the beginning, the right-hand sides of (4.3) and (4.4) as being known. Then, we introduce the complex-variable function:

$$F(z) = \Gamma - i\gamma + \sum_{n=1}^{\infty}(A_n - iB_n)z^n, \tag{4.6}$$

defined in the domain $|z| < 1$. Then, the dual series equations (4.3) and (4.4) can be written in the form

$$\begin{cases} \mathrm{Re}\{F(z)\} = f_j(\phi) & \text{for } z \in (a_j, b_j), \quad j = 1, \ldots, N, \\ \mathrm{Im}\{F(z)\} = g_j(\phi) & \text{for } z \in (b_j, a_{j+1}), \quad j = 1, \ldots, N. \end{cases} \tag{4.7}$$

The relations (4.7) yield the boundary values along the unit circle of the real part (along the strips) and the imaginary part (along the slots) of the analytic function $F(z)$. The problem of determining the function $F(z)$ by means of these conditions is known as a Volterra problem [18], [10]. In order to obtain the solution $F(z)$ we consider the auxiliary function

$$H(z) = \prod_{j=1}^{N}\left(e^{-i\frac{\beta_j - \alpha_j}{4}}\sqrt{\frac{z - b_j}{z - a_j}}\right),$$

where

$$\left(\sqrt{\frac{z - b_j}{z - a_j}}\right)_{z=0} = e^{i\frac{\beta_j - \alpha_j}{2}}. \tag{4.8}$$

Along the unit circle we have

$$H(z') = \begin{cases} |H(z')| & \text{on the arc } (b_j, a_{j+1}), \\ i \cdot |H(z')| & \text{on the arc } (a_j, b_j). \end{cases} \tag{4.9}$$

We now define a new unknown function

$$G(z) = \frac{F(z)}{iH(z)}. \tag{4.10}$$

Along the strips we have $\mathrm{Re}\{G(z')\} = -\frac{\mathrm{Re}\{R(z')\}}{|H(z')|}$ and along the slots $\mathrm{Re}\{G(z')\} = \frac{\mathrm{Im}\{F(z')\}}{|H(z')|}$. Then, (4.7) enables us to determine the values of the real part of the function $G()$ along the whole unit circle

$$\mathrm{Re}\{G(z')\} = -|H(z')|^{-1} \cdot f_j(\phi)$$
$$\text{for } z' = e^{i\phi} \in \mathrm{arc}(a_j, b_j), \tag{4.11}$$
$$\mathrm{Re}\{G(z')\} = |H(z')|^{-1} . g_j(\phi),$$
$$\text{for } z' = e^{i\phi} \in \mathrm{arc}(b_j, a_{j+1}). \tag{4.12}$$

This is a Dirichlet boundary-value problem for the real part of the complex function $G(z)$. The complex function $G(z)$ can be explicitated by means of Schwartz's formula [18]–[20]; this yields the function $F(z) = iH(z)G(z)$:

$$F(z) = \frac{H(z)}{\pi} \cdot \left\{\sum_{j=1}^{N} \int_{\overline{a_j b_j}} \frac{f_j(\phi)}{|H(z')|(z' - z)} dz' + \sum_{j=1}^{N} \int_{\overline{b_j a_{j+1}}} \frac{g_j(\phi)}{|H(z')|(z' - z)} dz' + \frac{\Gamma + i\gamma}{\overline{H(0)}}\right\}, \tag{4.13}$$

where $\overline{a_j b_j}$ and $\overline{b_j a_{j+1}}$ are symbols for the $\mathrm{arc}(a_j, b_j)$ and $\mathrm{arc}(b_j, a_{j+1})$; $\overline{H(0)}$ is the complex conjugate of $H(0)$; and ϕ is defined as in (4.11) or (4.12).

The unknown coefficients A_n, B_n are in fact the Mac Laurin expansion coefficients of the functions $F(z)$. The infinite linear system for their determination results by matching the coefficient of $z^n (n = 1, 2, \ldots)$ on the two sides of relation (4.13). We shall write this system in the matrix form

$$\mathbf{R}\vec{\mathbf{x}} = \mathbf{S}\vec{\mathbf{v}} + \mathbf{T}\vec{\mathbf{q}}, \tag{4.14}$$

where

$$\vec{\mathbf{x}}^t = (A_1, B_1, A_2, B_2, \ldots),$$
$$\vec{\mathbf{v}}^t = (V_1, \ldots, V_{\mathrm{ne}}),$$
$$\vec{\mathbf{q}}^t = (q_1, \ldots, q_{\mathrm{ne}}). \tag{4.15}$$

The linear system (4.14) is explicitated in the Appendix.

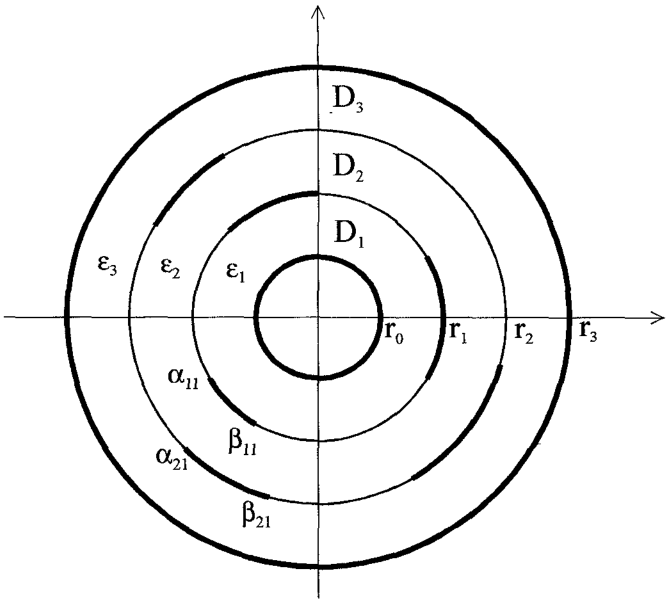

Fig. 4. Cross-section of a three-layer (two-surface) cylindrical multiconductor line.

The function $F(z)$ has a definite physical meaning in the case $\eta(n) = 0$ (i.e. $r_1 \to 0, r_2 \to \infty$): it is just the complex potential function $F(z) = V + iU$. Therefore, it must be finite along the unit circle. Thus, to compensate for the singularities of the function $H(z)$ in the a_k points, the expression in braces in (4.13) must vanish in all these points. Hence, we obtain the existence conditions:

$$\frac{\Gamma + i\gamma}{H(0)} - \sum_{j=1}^{N} \frac{1}{\pi} \int_{a_j b_j} \frac{f_j(\phi)}{|H(z')|(z' - a_k)} dz' + \sum_{j=1}^{N} \frac{1}{\pi} \int_{b_j a_{j+1}} \frac{g_j(\phi)}{|H(z')|(z' - a_k)} dz' = 0\,, \quad k = 1, \ldots, N. \tag{4.16}$$

The coefficient of z^0 in the Mac Laurin expansion must be equal to $F(0) = \Gamma - i\gamma$; this yields an additional compatibility condition besides the relations (4.16). All the $N + 1$ compatibility conditions have the same imaginary part, which is in fact an identity. We eliminate the constant γ and finally the compatibility conditions (4.16) can be written in matrix form

$$\mathbf{D}\vec{\mathbf{x}} = \mathbf{A}\vec{\mathbf{v}} + \mathbf{B}\vec{\mathbf{q}} \tag{4.17}$$

Again, the linear system (4.17) is explicitly given in the Appendix.

If we eliminate the unknown infinite vector $\vec{\mathbf{x}}$ between the (4.14) and (4.17) we finally obtain the capacitance matrix of the given system:

$$\mathbf{C} = \left(\mathbf{D} \cdot \mathbf{R}^{-1} \cdot \mathbf{T} - \mathbf{B}\right)^{-1} \cdot \left(\mathbf{A} - \mathbf{D} \cdot \mathbf{R}^{-1} \cdot \mathbf{S}\right). \tag{4.18}$$

This relation gives an exact expression of the capacitance matrix. To obtain a numerical estimation we must truncate the infinite matrices. The good convergence of the method is assured by the function $\eta(n)$ which enters into the terms of the matrix $\mathbf{D}$.

V. The Case of Two Circuit Surfaces

We outline now how the method applies to the case where more circuit surfaces are present. For the sake of simplicity, we only consider a line with two circuit surfaces; the general case can be dealt with in a similar way. The structure is shown in Fig. 4. It consists of four surfaces S_0, S_1, S_2, S_3 of radii r_0, r_1, r_2, r_3, respectively. The cylindrical surfaces S_0 and S_4 are grounded and the surfaces S_1 and S_2, separate regions D_1, D_2, D_3 filled with different dielectrics (of relative dielectric constants $\varepsilon_1, \varepsilon_2$and ε_3). On the circuit surface S_1 are placed $N1$ conducting strips characterized by angles α_{1j}, β_{1j} ($j = 1, \ldots, N1$) while on the S_2 circles are placed $N2$ conducting strips of angular abscissa α_{2j}, β_{2j} ($j = 1, \ldots, N2$). We denote by $V^{(j)}$the potential function in the domain D_j. Then, we have

$$V^{(1)}(r,\phi) = \Gamma^{(1)} R_0^-(r, r_0, r_1) + \sum_{n=1}^{\infty} R_n^-(r, r_0, r_1)(A_n'^{(1)} \cos n\phi + B_n'^{(1)} \sin n\phi) \qquad r_0 \le r \le r_1, \tag{5.1}$$

$$\begin{aligned} V^{(2)}(r,f) &= \Gamma^{(2)} R_0^-(r, r_1, r_2) + \sum_{n=1}^{\infty} R_n^-(r, r_1, r_2)\left(A_n'^{(2)} \cos n\phi + B_n'^{(2)} \sin n\phi\right) \\ &= \Gamma^{(1)} R_0^-(r, r_2, r_1) + \sum_{n=1}^{\infty} R_n^-(r, r_2, r_1)\left(A_n'^{(1)} \cos n\phi + B_n'^{(1)} \sin n\phi\right) \qquad r_1 \le r \le r_2, \end{aligned} \tag{5.2}$$

$$V^{(3)}(r,\phi) = \Gamma^{(2)} R_0^-(r, r_3, r_2) + \sum_{n=1}^{\infty} R_n^-(r, r_3, r_2)\left(A_n'^{(2)} \cos n\phi + B_n'^{(2)} \sin n\phi\right) \qquad r_2 \le r \le r_3. \tag{5.3}$$

These expressions ensure that the potential function vanishes on the grounded surfaces and is continuous in the whole structure. Along the circuit surface we also have:

$$\begin{aligned} &\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} U^{(2)}(r,\phi) - \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} U^{(1)}(r,\phi) \\ &= \sum_{n=1}^{\infty} [1 + \delta_{11}(n)]\left(B_n^{(1)} \cos n\phi - A_n^{(1)} \sin n\phi\right) + \sum_{n=1}^{\infty} \delta_{12}(n)\left(B_n^{(2)} \cos n\phi - A_n^{(2)} \sin n\phi\right) - \frac{Q^{(1)}}{2\pi(\varepsilon_1+\varepsilon_2)}\phi, \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\frac{\varepsilon_3}{\varepsilon_2+\varepsilon_3} U^{(3)}(r,\phi) - \frac{\varepsilon_2}{\varepsilon_2+\varepsilon_3} U^{(2)}(r,\phi) \\ &= \sum_{n=1}^{\infty} \delta_{21}(n)\left(B_n^{(1)} \cos n\phi - A_n^{(1)} \sin n\phi\right) + \sum_{n=1}^{\infty} [1 + \delta_{22}(n)]\left(B_n^{(2)} \cos n\phi - A_n^{(2)} \sin n\phi\right) - \frac{Q^{(2)}}{2\pi(\varepsilon_2+\varepsilon_3)}\phi, \end{aligned} \tag{5.5}$$

where

$$\delta_{jj}(n) = \frac{2\varepsilon_j}{\varepsilon_j+\varepsilon_{j+1}} \frac{r_{j-1}^{2n}}{r_j^{2n}-r_{j-1}^{2n}} + \frac{2\varepsilon_{j+1}}{\varepsilon_j+\varepsilon_{j+1}} \frac{r_j^{2n}}{r_{j+1}^{2n}-r_j^{2n}}, \quad (j=1,2), \tag{5.6}$$

$$\delta_{12}(n) = \frac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2} \frac{r_1^n \cdot r_2^n}{r_1^{2n}-r_2^{2n}}, \ \delta_{21} = \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_2+\varepsilon_3}\delta_{12}, \tag{5.7}$$

$$\frac{Q^{(1)}}{2\pi} = \left(\frac{\varepsilon_2}{\ln\frac{r_2}{r_1}} + \frac{\varepsilon_1}{\ln\frac{r_1}{r_0}}\right)\Gamma^{(1)} - \frac{\varepsilon_2}{\ln\frac{r_2}{r_1}}\Gamma^{(2)},$$

$$\frac{Q^{(2)}}{2\pi} = \left(\frac{\varepsilon_3}{\ln\frac{r_3}{r_2}} + \frac{\varepsilon_2}{\ln\frac{r_2}{r_1}}\right)\Gamma^{(2)} - \frac{\varepsilon_2}{\ln\frac{r_2}{r_1}}\Gamma^{(1)}. \tag{5.8}$$

The remaining boundary conditions give the system of dual series equations for determining the coefficients A_n, B_n in the form

$$\Gamma^{(r)} + \sum_{n=1}^{\infty}\left(A_n^{(r)}\cos n\phi + B_n^{(r)}\sin n\phi\right) = V_j^{(r)} + \sum_{n=1}^{\infty}\left(\eta_{r1}(n)A_n^{(1)} + \eta_{r2}(n)A_n^{(2)}\right)\cos n\phi + \sum_{n=1}^{\infty}\left(\eta_{r1}(n)B_n^{(1)} + \eta_{r2}(n)B_n^{(2)}\right)\sin n\phi,$$
$$\phi \in (\alpha_{rj}, \beta_{rj}), \ (j=1,\ldots,Nr) \tag{5.9}$$

$$\sum_{n=1}^{\infty}\left(A_n^{(r)}\sin n\phi - B_n^{(r)}\cos n\phi\right) = (\varepsilon_r+\varepsilon_{r+1})^{-1}\left(\sum_{l=1}^{j} q_j^{(r)} - \frac{\phi-\alpha_1}{2\pi}\sum_{l=1}^{Nr} q_j^{(r)}\right) + \gamma^{(r)},$$
$$\text{for } \phi \in (\beta_{r,j}, \alpha_{r,j+1}), \ (j=1,\ldots,Nr) \tag{5.10}$$

In the above equations the superscript r takes the value $r=1$ for the circle S_1 and the value $r=2$ for the other circuit surface. The system of equations (5.9) and (5.10) is written in a form suitable for applying the method developed in Section IV; the singular part of the integral equation is separated in the left-hand side of this equations, where the functions $\eta_{11}(n), \ldots, \eta_{22}(n)$ are determined in terms of the radii $r_0, \ldots, r_3$ and of dielectric constants $\varepsilon_1, \varepsilon_2, \varepsilon_3$.

Finally, we obtain the compatibility conditions

$$\mathbf{D_{11}}\vec{\mathbf{x}}^{(1)} + \mathbf{D_{12}}\vec{\mathbf{x}}^{(2)} = \mathbf{A_{11}}\vec{\mathbf{v}}^{(1)} + \mathbf{A_{12}}\vec{\mathbf{v}}^{(2)} + \mathbf{B_{11}}\vec{\mathbf{q}}^{(1)} + \mathbf{B_{12}}\vec{\mathbf{q}}^{(2)}$$
$$\mathbf{D_{21}}\vec{\mathbf{x}}^{(1)} + \mathbf{D_{22}}\vec{\mathbf{x}}^{(2)} = \mathbf{A_{21}}\vec{\mathbf{v}}^{(1)} + \mathbf{A_{22}}\vec{\mathbf{v}}^{(2)} + \mathbf{B_{21}}\vec{\mathbf{q}}^{(1)} + \mathbf{B_{22}}\vec{\mathbf{q}}^{(2)} \tag{5.11}$$

and also two infinite systems of equations for determining the two sets of Fourier coefficients

$$\mathbf{R_{11}}\vec{\mathbf{x}}^{(1)} + \mathbf{R_{12}}\vec{\mathbf{x}}^{(2)} = \mathbf{S_{11}}\vec{\mathbf{v}}^{(1)} + \mathbf{S_{12}}\vec{\mathbf{v}}^{(2)} + \mathbf{T_{11}}\vec{\mathbf{q}}^{(1)} + \mathbf{T_{12}}\vec{\mathbf{q}}^{(2)}$$
$$\mathbf{R_{21}}\vec{\mathbf{x}}^{(1)} + \mathbf{R_{22}}\vec{\mathbf{x}}^{(2)} = \mathbf{S_{21}}\vec{\mathbf{v}}^{(2)} + \mathbf{S_{22}}\vec{\mathbf{v}}^{(2)} + \mathbf{T_{21}}\vec{\mathbf{q}}^{(1)} + \mathbf{T_{22}}\vec{\mathbf{q}}^{(2)} \tag{5.12}$$

Elimination of infinite unknown vectors $\vec{\mathbf{x}}^{(1)}, \vec{\mathbf{x}}^{(2)}$ between the systems (5.11) and (5.12) yields the desired Maxwell capacitance matrix of the structure.

VI. Multilayered Multiconductor Structures

In order to show how our method applies in the case of multilayered multiconductor structures we consider the case of strips on a suspended substrate. The geometry of the problem is given in Fig. 4. The domains D_1, D_3 are filled with air ($\varepsilon_1 = \varepsilon_3 = 1$) and the strips are placed on the surface S_2. In this case we shall consider again the expression of the potential given in Section V. There are no charges along the circle S_1, hence the relation (A2.7) yields:

$$(1+\delta_{11}(n))\left(A_n^{(1)} - iB_n^{(1)}\right) + \delta_{12}(n)\left(A_n^{(2)} - iB_n^{(2)}\right) = 0, \tag{6.1}$$

$$Q^{(1)} = 0. \tag{6.2}$$

The relations (6.1) and (6.2) determine the coefficients $A_n^{(1)} - iB_n^{(1)}$ in terms of the coefficients $A_n^{(2)} - iB_n^{(2)}$ and also coefficient $\Gamma^{(1)}$ as a function of $\Gamma^{(2)}$. Finally, the boundary conditions along the circuit surface S_2 take the form (4.3) and (4.4) where

$$\delta(n) = \delta_{22}(n) - \frac{\delta_{12}(n)\cdot\delta_{21}(n)}{1+\delta_{11}(n)}, \tag{6.3}$$

$$\varepsilon^* = \frac{1}{\varepsilon_2+\varepsilon_3}\left(\frac{\varepsilon_3}{\ln\frac{r_3}{r_2}} + \frac{\varepsilon_2}{\ln\frac{r_2}{r_1}} - \varepsilon_2^2\left(\frac{\varepsilon_1}{\ln\frac{r_1}{r_0}} + \frac{\varepsilon_2}{\ln\frac{r_2}{r_1}}\right)^{-1}\right). \tag{6.4}$$

The solution is obtained by the relations given in Section IV for the above values of parameters. The general case of multilayered structures can be treated in the same way; in fact, the solution of the problem is reduced to the one for a single layer, but with proper coefficients $\delta(n)$ end ε^*.

VII. Numerical Results

The theory developed in the previous sections was implemented into a computer program written in the MATHCAD language running on a PC-AT 486DX. The clock frequency was 25 MHz and the program operates with 15 significant digits. We applied this program to various test problems.

1) In order to compare the results given by this method with some exact analytical formulae we considered the case of a single strip on a cylindrical stripline-like microstrip transmission line. In this case analytical formulae are available for capacitances expressed in terms of elliptic functions [11]. We found very good agreement in both

cases. Thus, for example, if we take

$$h = 1.477575743, \quad r_1 = e^{-h}, \quad r_2 = e^h,$$
$$\alpha_2 = -\beta_1 = \pi/3, \quad \beta_2 = -\alpha_1 = 2\pi/3,$$

the method gives

$$C(1,1) = 15.795598577, \quad C(1,2) = -1.302715389,$$

by considering only 6 terms A_n, B_n. These results coincide within 10 digits with those given by the analytical formulae.

2) In the case of a symmetrical strip of 2c-widths, inside a $L \times (h_1 + h_2)$ rectangular box we compared the results given by our method with the results obtained in [12] on the basis of an analytical approach based on Carleman-type singular integral equation. These results are also compared with the values obtained previously [16] by using two or three different methods: transverse modal analysis (TMA), Potential Theory (PT), method of moments (MM), and approximate conformal mapping (ACM). The characteristic impedance is given in Table I for various values of geometrical and electrical parameters.
The results are in good agreement with those obtained previously. Due to the fact that we must consider on the circle a structure with two strips, the number of coefficients involved in our analysis is about 20.

3) *Comparison with the spectral-domain method:* A number of planar structures were analysed with the present method, and the results compared with those obtained from a spectral-domain method quasi-TEM technique for multiconductor lines on multilayered dielectric substrates [5], [6]. The implementation described in [6] makes use of edge-singular basis functions for the strip charge density; the spectral-domain superposition integrals are evaluated by turning them into discrete sum, which are then truncated according to a relative convergence principle. The spectrum is discretized by imposing periodic boundary conditions or by considering a line laterally bounded by electric or magnetic walls. As a first set of test structures, three four-conductor coupled microstrip lines were chosen with uniform spacing $s$and strip width w; periodic boundary conditions were imposed, with the line centered into a box of total width $4\,w + 5\,s$. The substrate dielectric constant is $e_r = 9$ and the substrate thickness is h. The line dimensions were chosen as follows:

Structure 1: $s/h = (\pi/2 - 2/3)/\ln 3 \approx 0.823;$
$w/h = 2/3 \ln 3 \approx 0.607$

Structure 2: $s/h = (\pi/2 - 1/2)/\ln 2 \approx 1.545;$
$w/h = 1/2 \ln 2 \approx 0.721$

Structure 3: $s/h = (\pi/2 - 1/3)/\ln 3/2 \approx 3.052;$
$w/h = 1/3 \ln 3/2 \approx 0.822$

Thus, from structure 1 to structure 3 the coupling between strips decreases and the coupling to the ground plane increases. Since the structure is strictly periodic, the capacitance element $C_{i,j}$ only depends on $|i - j|$. The results obtained are summarized in Table II; only the first row of the capacitance matrix is shown. The spectral-domain results were computed with 10 basis functions per strip and a number of spatial frequency samples related to the relative convergence criterion. The agreement between the two methods is fairly good both for the tightly coupled and for the loosely coupled line; however, the accuracy achieved by spectral-domain method in estimating the coupling between distant lines may be critical.

In order to see how the method works in the case of a line with a large number of strips we consider a cylindrical microstrip line with 32 equally spaced strips. The strips has the same central angle and this is equal to the slot central angle. We take $r_1 = 0.5, \varepsilon_1 = 9, \varepsilon_2 = 1$.

The same structure was analysed through the spectral domain method by first turning it into a planar one by means of conformal mapping. The results obtained from the two methods are reported by showing the first row only of the capacitance matrix; for $C(1,j), 16 < j < 32$ one must remember that $C(1,j) = C(1, 34 - j), \quad j = 18 \ldots 32$. The result in brackets was derived from the spectral domain method; the other from the present approach. One has

$$C(1,1) = 12.909[12.953]$$
$$C(1,2) = -4.074[-4.066]$$
$$C(1,3) = -0.695[-0.678]$$
$$C(1,4) = -0.233[-0.218]$$
$$C(1,5) = -0.095[-0.083]$$
$$C(1,6) = -0.044[-0.034]$$
$$C(1,7) = -0.022[-0.014]$$
$$C(1,8) = -0.012[-0.006]$$
$$C(1,9) = -0.0077[-0.0024]$$
$$C(1,10) = -0.0055[-0.0011]$$
$$C(1,11) = -0.0042[-0.0004]$$
$$C(1,12) = -0.0035[-0.0002]$$
$$C(1,13) = -0.0031[-0.00005]$$
$$C(1,14) = -0.0028[-0.00005]$$
$$C(1,15) = -0.0026[+0.00001]$$
$$C(1,16) = -0.0025[-0.00003]$$
$$C(1,17) = -0.0025[+0.00002]$$

The computation with the spectral domain method was carried out with 5 basis functions per strip. Also in this case, the accuracy achieved by the spectral method becomes poor for loosely coupled strips (notice that for $j > 11$ the result from this spectral domain method is hardly meaningful, although the computation was performed in double precision). The above analysis required about 15 minutes CPU both for the MATHCAD implementation and for the FORTRAN implementation of the spectral domain method running on a VAX STATION 3100.

TABLE I
COMPARISON OF Z0 FOR A SINGLE STRIP WITH EXISTING VALUES

h_2/h_1	L/2c	$2c/h_1$	$\varepsilon_1/\varepsilon_2$	$Z_0(\Omega)$	$Z_0(\Omega)$ [12]	TMA [16]	PT [16]	MM [16]	ACM [16]
5	10	1	9.6	49.153	49.08	48.4	48.5	-	-
9	10	2	9.6	33.72	33.63	33.1	32.8	-	-
5	10	0.8	9.6	54.52	54.45	53.8	53.9	-	-
5	6	1	9.6	48.62	48.60	47.9	48.5	-	-
5	10	0.4	9.6	70.40	70.38	69.7	70.9	-	-
9	10	1	6.0	61.32	61.24	60.49	-	62.71	60.97
9	10	4	6.0	26.26	26.09	25.95	-	27.30	26.03
9	10	0.4	6.0	87.04	87.41	86.30	-	91.37	89.91

TABLE II
a: PRESENT APPROACH; b: SPECTRAL DOMAIN METHOD

Structure	Method	C_{11}	C_{12}	C_{13}	C_{14}
1	a	14.762	-1.866	-0.198	-1.866
1	b	14.792	-1.904	-0.224	-1.904
2	a	15.439	-0.731	-0.093	-0.731
2	b	15.430	-0.745	-0.106	-0.745
3	a	16.3645	-0.1509	-0.0378	-0.1509
3	b	16.369	-0.157	-0.044	-0.157

TABLE III
THE PROPAGATION CONSTANT FOR A CYLINDRICAL STRUCTURE WITH SIX STRIPS LOCATED ON TWO LAYERS

Separation angle γ (degrees)	Number of coefficients [17]	1-st mode	2-nd mode	3-rd mode	4-th mode	5-th mode	6-th mode
0.00	12	2.045162	2.194719	2.389983	3.073252	3.152087	3.403444
0.00	16	2.04508	2.194718	2.39001	3.073376	3.152093	3.403722
0.00	[17]	2.0722	2.1945	2.3896	3.0725	3.1504	3.4138
35.86	12	2.052133	2.234518	2.417062	2.686693	3.102286	3.354923
35.86	16	2.052069	2.234541	2.417061	2.686714	3.102287	3.35495
35.86	[17]	2.0768	2.2354	2.4215	2.6929	3.1001	3.3641
106.17	12	2.093422	2.275377	2.337481	2.430684	2.78487	2.996957
106.17	16	2.09335	2.2754	2.33738	2.430698	2.784814	2.996659
106.17	[17]	2.1075	2.2978	2.3393	2.4311	2.7887	3.0065

4) We applied the method developed in Section V to the cylindrical multiconductor transmission line having two layers of strips located at $r_1 = 2$ and $r_2 = 3$. Each of the two layers of strips consists in three strips of 10.195° central half angle and the strips on the same r circle are 20.39° apart. We denote by γ the separation angle between the strips ($\alpha_{2j} = \alpha_{1j} + \gamma, \beta_{2j} = \beta_{1j} + \gamma$; $j = 1, 2, 3$). This problem was considered in [17] by an iterative technique in spectral domain. In Table III we give the propagation constants obtained by applying the present method with 12 and 16, respectively, unknown coefficients on every layer and for three values of separation angle. For comparison the values given in [17] are also shown. It is to be noted that the present method and the results obtained in [17] are in very good agreement.

5) As the last example, we computed the characteristic impedance Z and the effective dielectric constant ϵ_{eef} for a pair of coupled strips on a suspended substrate inside a rectangular box. The cross-section of a boxed microstrip line is shown in Fig. 5. The geometry considered is $L = 5$ mm, $h_1 = 3.635$ mm, $h_2 = 0.635$ mm, $h_3 = 5$ mm, $2c = 1$ mm, the distance between strips is 0.1 mm, $\varepsilon_1 = \varepsilon_3 = 1, \varepsilon_2 = 9.6$. This problem was considered in [15] on a spectral domain formulation combined with some methods for speeding up the convergence.

We truncated the infinite series at 24 terms and in Table IV the obtained results are compared with those given in [15]. Again, very good agreement is found between the present method and the results given in [15].

VIII. CONCLUSION

A new method for determining the Maxwell capacitance matrix of a coupled multilayer multiconductor microstrip

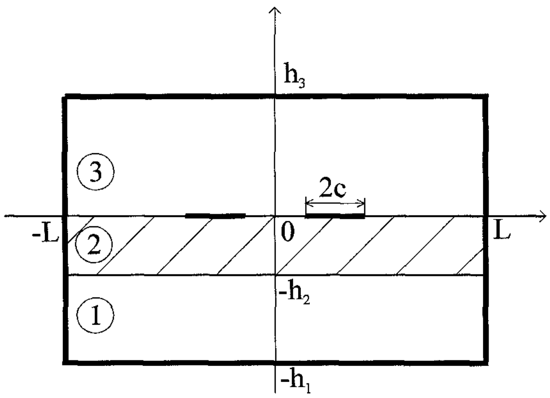

Fig. 5. Boxed suspended coupled strips.

TABLE IV
CHARACTERISTIC IMPEDANCE AND EFFECTIVE DIELECTRIC CONSTANT FOR SUSPENDED COUPLED STRIPS

	Present method	Results given in [15]
Z_{odd}	30.835936 Ω	30.8360 Ω
Z_{even}	182.87994 Ω	182.87994 Ω
$\varepsilon_{eff.odd}$	4.608939	4.608930
$\varepsilon_{eff.even}$	2.1366193	2.136619

line is given. The method applies to the case of cylindrical structures as well as to the case of planar lines. The method is based on solving the singular part of the coupled series equations of the problem by means of a Volterra boundary-value problem; the rigorous solution is expressed by means of some infinite matrices that have very good convergence properties. Applications are shown to several structures: isolated and coupled strips inside rectangular boxes, multistrip structures, suspended lines, and cylindrical structures with two layers of strips. The method allows the capacitance matrix to be evaluated with very high accuracy and is computationally efficient also for the case of multistrip and multilayered lines.

APPENDIX

We give in this Appendix the computation relations involved in Section IV. The Mac Laurin expansion of the function $F(z)/H(z)$ must match the corresponding coefficients resulting in the expansion of the brace in (4.13). We get for $m = 0$:

$$\begin{aligned}\frac{\Gamma - i\gamma}{H(0)} &= -\sum_{j=1}^{N} \mathrm{AC}(0,j)V_j + (\varepsilon_1+\varepsilon_2)\\ &\times \sum_{j=1}^{N}\left(\mathrm{BC}(0,j)\sum_{l=1}^{j} q_l - \mathrm{BCS}(0,j)\sum_{l=1}^{N} q_l\right)\\ &- \sum_{n=1}^{\infty}\eta(n)[(A_n - iB_n)K_2(n) + (A_n+iB_n)K_1(n)]\\ &+ \frac{\overline{\Gamma + i\gamma}}{\overline{H(0)}}, \end{aligned} \tag{A.1}$$

and for $m = 1, 2, \ldots$

$$\begin{aligned}&-2(\Gamma - i\gamma)K_1(m) - 2\sum_{n=1}^{m-1} K_1(m-n)(A_n - iB_n)\\ &+ \frac{1}{H(0)}(A_m - iB_m) = -\sum_{j=1}^{N}\mathrm{AC}(m,j)V_j\\ &+ (\varepsilon_1+\varepsilon_2)\sum_{j=1}^{N}\left(\mathrm{BC}(m,j)\sum_{l=1}^{j} q_l - \mathrm{BCS}(m,j)\sum_{l=1}^{N} q_l\right)\\ &- \sum_{n=1}^{\infty}\eta(n)[(A_n - iB_n)K_2(n-m) + (A_n + iB_n)\\ &\times K_1(n+m)] \end{aligned} \tag{A.2}$$

Here we have denoted

$$\begin{aligned}\mathrm{AC}(m,j) &= \frac{1}{\pi}\int_{a_j b_j} \frac{1}{|H(z')|}\frac{dz'}{z'^{m+1}}\\ &= \frac{i}{\pi}\int_{\alpha_j}^{\beta_j}\frac{e^{-\mathrm{im}\phi}}{|H(e^{i\phi})|}d\phi, \quad m = 0,1,\ldots \end{aligned}\tag{A.3}$$

$$\begin{aligned}\mathrm{BC}(m,j) &= \frac{1}{\pi}\int_{b_j a_{j+1}} \frac{1}{|H(z')|}\frac{dz'}{z'^{m+1}}\\ &= \frac{i}{\pi}\int_{\beta_j}^{\alpha_{j+1}}\frac{e^{-\mathrm{im}\phi}}{|H(e^{i\phi})|}d\phi, \quad m = 0,1,\ldots \end{aligned}\tag{A.4}$$

$$\begin{aligned}\mathrm{BCS}(m,j) &= \frac{1}{\pi}\int_{b_j a_{j+1}} \frac{\phi-\alpha_1}{\pi}\frac{1}{|H(z')|}\frac{dz'}{z'^{m+1}}\\ &= \frac{i}{\pi}\int_{\beta_j}^{\alpha_{j+1}}\frac{\phi-\alpha_1}{\pi}\frac{e^{-\mathrm{im}\phi}}{|H(e^{i\phi})|}d\phi, \quad m = 0,1,\ldots \end{aligned}\tag{A.5}$$

$$K_1(r) = \frac{1}{2\pi}\int_{C_e}\frac{z'^{-r-1}}{|H(z')|}dz', \quad r = 0,1,2,\ldots \tag{A.6}$$

$$K_2(r) = \frac{1}{2\pi}\int_{C_e}\frac{z'^{r-1}}{|H(z')|}dz', \quad r = 1,2,\ldots \tag{A.7}$$

In relations (A.6), (A.7) $C_e = \bigcup_{j=1}^{N}(a_j, b_j)$ i.e. the set of all strips; we shall also denote by C the unit circle.

Likewise, the compatibility conditions (4.16) can be written in the form

$$\begin{aligned}&\Gamma\cos\hat{\phi} - \gamma\sin\hat{\phi} - \sum_{j=1}^{N}A'(k,j)V_j + (\varepsilon_1+\varepsilon_2)^{-1}\\ &\times\sum_{j=1}^{N}\left(B'(k,j)\sum_{l=1}^{j}q_l - \mathrm{BS}'(k,j)\sum_{l=1}^{N}q_l\right)\\ &-\sum_{n=1}^{\infty}\eta(n)\{\mathrm{Re}\{I1(k,n) - I2(k,n)\}A_n\\ &-\mathrm{Im}\{I2(k,n) + I1(k,n)\}B_n\} = 0, \quad k = 1,2,\ldots,N \end{aligned}\tag{A.8}$$

Here we have denoted

$$H(0) = e^{i\hat{\phi}}, \tag{A.9}$$

$$A'(k,j) = \frac{1}{2\pi}\int_{\alpha_j}^{\beta_j} \frac{\text{ctg}\frac{\phi-\alpha_1}{2}}{|H(e^{i\phi})|}d\phi, \tag{A.10}$$

$$B'(k,j) = \frac{1}{2\pi}\int_{\beta_j}^{\alpha_j+1} \frac{\text{ctg}\frac{\phi-\alpha_1}{2}}{|H(e^{i\phi})|}d\phi, \tag{A.11}$$

$$BS'(k,j) = \frac{1}{2\pi}\int_{\beta_j}^{\alpha_j+1} \frac{\phi-\alpha_1}{2\pi}\frac{\text{ctg}\frac{\phi-\alpha_1}{2}}{|H(e^{i\phi})|}d\phi, \quad (k,j=1,\ldots,N), \tag{A.12}$$

$$I1(k,n) = \frac{1}{2\pi}\int_{C_e} \frac{z'^{-n}}{|H(z')|(z'-a_k)}dz', \quad (k=1,\ldots N;\ n=1,2,\ldots), \tag{A.13}$$

$$I2(k,n) = -\frac{1}{2\pi}\int_{C_e} \frac{z'^{n}}{|H(z')|(z'-a_k)}dz', \quad (k=1,\ldots,N;\ n=0,1,\ldots). \tag{A.14}$$

The numerical evaluation of the integrals (A.3–A.5) and (A.10–A.12) can be performed by using a Chebyshev-type integration formula

$$\frac{1}{\pi}\int_{\alpha}^{\beta} F(\phi)d\phi = \frac{\beta-\alpha}{2m}\sum_{j=1}^{m}\sin\frac{(2j-1)\pi}{2m} \times F\left(\frac{\beta-\alpha}{2}\cos\frac{(2j-1)\pi}{2m}+\frac{\beta+\alpha}{2}\right). \tag{A.15}$$

The complex integrals (A.6–A.7) and (A.13–A.14) along the set C_e of the strips can be replaced by integrals along a circle C_2 of radius greater than 1, or along a circle C_1 inside the unit circle. Thus, for example, we have by means of Cauchy's theorem

$$\oint_{C_2} \frac{z'^n}{H(z')(z'-a_k)}dz' = -\frac{1}{i}\int_{C_e}\frac{z'^n}{|H(z')|(z'-a_k)}dz' + \int_{C-C_e}\frac{z'^n}{|H(z')|(z'-a_k)}dz', \tag{A.16}$$

$$\oint_{C_1} \frac{z'^n}{H(z')(z'-a_k)}dz' = \frac{1}{i}\int_{C_e}\frac{z'^n}{|H(z')|(z'-a_k)}dz' + \int_{C-C_e}\frac{z'^n}{|H(z')|(z-a_k)}dz'. \tag{A.17}$$

Due to residue theorem the integral along the C_2 circle vanishes for $n < 0$ and the integral along the C_1 contour is zero for $n \geq 0$. Hence, the difference of the relations (A.16) and (A.17) yields

$$\oint_{C_e} \frac{z'^n}{|H(z')|(z'-a_k)}dz' = \frac{1}{2i}\oint_{C_2}\frac{z'^n}{H(z')(z'-a_k)}dz', \quad n=0,1,\ldots \tag{A.18}$$

and

$$\oint_{C_e} \frac{z'^{-n}}{|H(z')|(z'-a_k)}dz' = -\frac{1}{2i}\oint_{C_2}\frac{z'^{-n}}{H(z')(z'-a_k)}dz', \quad n=1,2,\ldots \tag{A.19}$$

The integrals along the circles C_1 and C_2 can be computed by using the trapezoidal rule and the Fast Fourier Transform algorithm.

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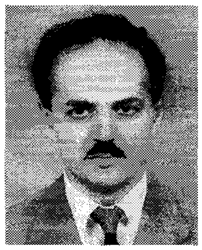

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